GEOMETRIC QUANTIZATION ON SYMPLECTIC FIBER BUNDLES

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Abstract

Consider a fiber bundle in which the total space, the base space and the fiber are all symplectic manifolds. We study the relations between the quantization of these spaces. In particular, we discuss the geometric quantization of a vector bundle, as oppose to a line bundle, over the base space that recovers the standard geometric quantization of the total space.

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I. Introduction.

In [1], the authors considered a family of symplectic manifolds and gave a topological condition under which these manifolds can be quantized simultaneously using the procedures of geometric quantization [2] [3]. More precisely, let $E \to B$ be a differentiable fiber bundle each fiber E_b is equipped with a symplectic form Ω_b so that $\{E_b, \Omega_b\}$ is symplectomorphic to the model fiber $\{F, \Omega_F\}$, with structure group of this bundle preserving the symplectic form. A closed two-form Ω_E on E is termed "closed extension" of the family of symplectic forms Ω_b if $\Omega_E = \Omega_b$ when restricted to the fiber. The authors showed that closed extension exists if it exists at the cohomology level. In which case, the line bundle $\mathscr{L} \to E$ whose curvature is Ω_E pulls back to a prequantization line bundle $\mathscr{L}_b \to E_b$. This then allows one to carry out the necessary calculations on $\mathscr{L}$ as a way to quantize the whole family of symplectic manifolds $\{E_b\}_{b\in B}$.

We are interested in the situation where the total space E, the base space B and the fiber F are all symplectic manifolds, and we want to study the relation between the quantization of these spaces. It is known that if the bundle $E \to B$ is trivial, then the relation between the quantum Hilbert spaces is given by the tensor product: $\mathcal{Q}(E) = \mathcal{Q}(F) \otimes \mathcal{Q}(B)$, where $\mathcal{Q}(X)$ denotes the quantum Hilbert space of the classical phase space X, suppressing in our notation the dependence on the polarizations chosen for the quantization *etc*.

Suppose there is a polarization on F that is preserved by the structure group of the fiber bundle, then there is an associated vector bundle $\mathcal{V} \to B$ whose fiber is $\mathcal{Q}(F)$. Our goal is to give conditons and procedures for quantizing this "prequantization vector bundle" that parallels the standard geometric quantization on line bundles. The resulting wavefunctions, *i.e.*, sections on this vector bundle covariant constant along a polarization on B, are vector-valued wavefunctions. This will provide a nice setup for multicomponent WKB method, since the quantization of E has both a scalar version ($\mathscr{L} \to E$) and a vector version ($\mathcal{V} \to B$). Multicomponent WKB theory has gain recent interest starting with the work of [4].

A detail setup for the quantization of vector bundles will be discussed in section 2, where we state our main theorem. In section 3 we show that the case of a particle moving under the influence of an external Yang-Mills field fits into our fiber bundle formality. This represents a non-trivial example where the whole system can be quantized, and the resulting Hilbert space is a twisted tensor product induced by the Yang-Mills potential. The prove of the theorem will occupy section 4.

II. The Setup.

Let $\{E, \Omega_E\}$, $\{B, \Omega_B\}$ and $\{F, \Omega_F\}$ be symplectic manifolds, we assume there are canonical one–forms α_B, α_E, α_F such that $\mathrm{d}\alpha_B = \Omega_B$, $\mathrm{d}\alpha_E = \Omega_E$ and $\mathrm{d}\alpha_F = \Omega_F$. Let $\mathcal{P}(F)$ be a polarization on F. Let $\pi : E \to B$ be a symplectic fiber bundle with fiber F, which we assume to be compact and simply connected. We further assume the structure group preserves the canonical one–form and the polarization. More precisely, there exist a local trivilization $\chi_i : U_i \times F \to E$ such that firstly, if $b \in U_i$, then $\chi_{i,b} = \chi_i(b, -) : F \to E_b$ satisfies $\chi^*_{i,b}(\alpha_E|E_b) = \alpha_F$. So Ω_E is a closed extension of the symplectic forms on the fibers E_b in the sense of [1]. Secondly, if $b \in U_i \cap U_j$, then $\chi_{ij,b} = \chi^{-1}_{j,b}\chi_{i,b} : F \to F$ preserves the polarization $\chi_{ij,b*}\mathcal{P}(F) = \mathcal{P}(F)$.

As a closed extension, the symplectic form Ω_E defines an Ehresmann connec-

tion on the bundle $E \to B$ so that for $e = \chi_i(b,f)$, $u \in T_eE$ is horizontal if $\Omega_E(u, \chi_{i,b*}\xi) = 0$ for all $\xi \in T_fF$. Note that the Ehresmann connection is defined up to a two–form on B. We denote by $v^\# \in T_eE$ the horizontal lift of $v \in T_bB$, and by $\mathfrak{hor}\,(B)$ the set of horizontal vector fields. We make the following observation:

Proposition 1. *Let $\alpha_\nabla = \alpha_E - \pi^*\alpha_B$, and let $\alpha^i = \chi_i^*\alpha_\nabla$ be the one–form on $U_i \times F$, v a vector field on U_i, then*

$$v^\# = v - \mathscr{H}_{\langle \alpha^i, v\rangle} \tag{1}$$

where we treat $w = \langle \alpha^i, v\rangle$ as a function on F with b fixed, and $\mathscr{H}_w$ is the Hamiltonian vector field on F defined through the equation $\Omega_F(\mathscr{H}_w, -) = -d_Fw$.

Proof. We must show that $\chi_i^*\Omega_E(v - \mathscr{H}_{\langle\alpha^i,v\rangle}, \xi) = 0$ for all vector fields v on B and ξ on F. Note that

$$\Omega_F(\mathscr{H}_w, \xi) = -\xi(w) = \mathscr{H}_w\langle\alpha_F, \xi\rangle - \xi\langle\alpha_F, \mathscr{H}_w\rangle - \langle\alpha_F, [\mathscr{H}_w, \xi]\rangle. \tag{2}$$

Using the facts that $[v,\xi] = 0$ and $v\langle\alpha^i,\xi\rangle = v\langle\alpha_F,\xi\rangle = 0$, we compute

$$\begin{aligned} \mathrm{d}\alpha^i(v - \mathscr{H}_w, \xi) &= (v - \mathscr{H}_w)\langle\alpha^i,\xi\rangle - \xi\langle\alpha^i, v - \mathscr{H}_w\rangle - \langle\alpha^i, [v - \mathscr{H}_w, \xi]\rangle \\ &= v\langle\alpha_F,\xi\rangle - \mathscr{H}_w\langle\alpha_F,\xi\rangle - \xi\langle\alpha^i, v\rangle + \xi\langle\alpha_F,\mathscr{H}_w\rangle + \langle\alpha_F, [\mathscr{H}_w,\xi]\rangle \\ &= -\xi\langle\alpha^i, v\rangle + \xi(w) = 0, \qquad \text{since } w = \langle\alpha^i, v\rangle. \quad \square \end{aligned}$$

Suppose we have a polarization $\mathcal{P}(E)$ on E that is composed of $\mathcal{P}(F)$ and horizontal vector fields. More precisely, let $e = \chi_i(b,f)$, $u \in \mathcal{P}_e(E)$ implies there is a $u_F \in \mathcal{P}_f(F)$ and a $u_B \in T_bB \otimes \mathbf{C}$ so that $u = \chi_{i,b*}u_F + u_B^\#$. So by abuse of notation we consider $\mathcal{P}(F)$ also as a foliation on E via the trivializing maps χ_i, our assumption that χ_{ij} preserves $\mathcal{P}(F)$ implies this push forward is independent on the coordinate patch we choose. And we denote the polarization $\mathcal{P}(E) = \mathcal{D}^\#(B) \oplus \mathcal{P}(F)$, where $\mathcal{D}(B)$ is a distribution on B and $\mathcal{D}^\#(B)$ its horizontal lift.

Let $\mathscr{L}_F$ be a prequantization line bundle on F and denote by $\mathcal{Q}(F)$ the space of sections on $\mathscr{L}_F$ covariant constant along the polarization $\mathcal{P}(F)$, and we will treat $\mathcal{Q}(F)$ as the quantum Hilbert space for the quantization of F, ignoring potential complications that may arise from the half-form bundle that appears the process of geometric quantization. Since F is assumed compact, $\mathcal{Q}(F)$ is finite dimensional, we fix a basis $\{\phi_1, \ldots, \phi_n\}$. We denote the quantization of the symplectomorphisms $\chi_{ij,b}$ by $\mathbf{X}_{ij,b} : \mathcal{Q}(F) \to \mathcal{Q}(F)$. Let $\mathcal{V} \to B$ be the unitary vector bundle whose transition functions given by $\mathbf{X}_{ij,b}$ and denote by $\Gamma(\mathcal{V})$ the space of sections.

Let $\mathscr{L}$ be a prequantization line bundle over E which pulls back to $\mathscr{L}_F$ by $\chi_{i,b}$. That is, there is a bundle map so that the diagram

$$\begin{array}{ccc} \mathscr{L}_F & \longrightarrow & \mathscr{L}_b \\ \big\downarrow & & \big\downarrow \\ F & \longrightarrow & E \end{array}$$

commutes. This will be the case if the connection one form α_E on E pulls back to the connection form α_F on F via $\chi_{i,b}$, as we have assumed. Since by assumption the transition functions $\chi_{ij,b}$ preserve both the connection one–form and the polarization, their quantization is by substitution:

$$\mathbf{X}_{ij,b}\phi(f) = \phi(\chi_{ij,b}(f)). \tag{3}$$

Denote by $\Gamma_{\mathcal{P}(E)}(\mathscr{L})$ the space of sections on $\mathscr{L}$ covariant constant along the polarization $\mathcal{P}(E)$, and by $\Gamma_{\mathcal{P}(F)}(\mathscr{L})$ the space of sections covariant constant along the foliation $\mathcal{P}(F)$. Let $\psi \in \Gamma_{\mathcal{P}(F)}(\mathscr{L})$, when restricted to a local trivilization patch $U_i \times F$, $\psi(\chi_i(b,f))$ must be of the form $\Psi^i_\mu(b)\phi_\mu(f)$. (We will frequently suppress the index i if no confusion will arise.) Thus we have established an isomorphism $\Gamma_{\mathcal{P}(F)}(\mathscr{L}) \to \Gamma(\mathcal{V})$. Consider the composition

$$\Gamma_{\mathcal{P}(E)}(\mathscr{L}) \to \Gamma_{\mathcal{P}(F)}(\mathscr{L}) \to \Gamma(\mathcal{V}), \tag{4}$$

the image of $\Gamma_{\mathcal{P}(E)}(\mathscr{L})$ defines a subspace $\Gamma_c(\mathcal{V})$ in $\Gamma(\mathcal{V})$ that plays the role of covariant constant sections. One would like to have a connection on the vector bundle $\mathcal{V}$ in order to define this notion of covariant constant section on $\mathcal{V}$. For technical reasons, we need to assume that

$$[\mathfrak{hor}\,(B), \mathcal{P}(F)] \subseteq \mathcal{P}(F). \tag{5}$$

where $[,]$ denotes vector field commutator.

Let $H : F \to \mathbf{R}$, we denote the prequantization operator [2]

$$\mathscr{O}(H) = -i\mathscr{H}_H - \langle \alpha_F, \mathscr{H}_H \rangle + H. \tag{6}$$

Through out this paper we will let $\mathscr{H}_H$ to denote the Hamiltonian vector field on F with respect to the symplectic form Ω_F, and the b dependence that H may have will be treated as constant in this regard. And $\langle , \rangle$ is the pairing between differential forms and vector fields, the inner product on $\mathcal{Q}(F)$ is denoted by $\ll \mid \gg$. Let U be the unitary group of the (finite-dimensional) Hilbert space $\mathcal{Q}(F)$ and $\mathfrak{u}$ its Lie algebra, define

$$\begin{aligned} &A^i(b) : T_b B \to \mathfrak{u} \\ &\left[A^i(b)v\right]_{\mu\nu} = i \ll \phi_\nu \mid \mathscr{O}\left(\langle \chi_i^* \alpha_\nabla, v\rangle\right) \phi_\mu \gg \end{aligned} \tag{7}$$

for all $b \in U_i$. We now summarize our assumptions and state our main theorem:

Theorem. *Let $E \to B$ be a symplectic fiber bundle with fiber F and transition functions $\chi_{ij,b} : F \to F$ preserving the connection one–form α_F and polarization $\mathcal{P}(F)$. Let $\Omega_E = d\alpha_E$ be the symplectic form on E that is a closed extension of Ω_F on the fibers. Suppose the Ehresmann connection defined by Ω_E satisfies (5). Then A^i in (7) defines a connection on the vector bundle $\mathcal{V} \to B$. Let $\mathcal{P}(E) = \mathcal{D}^{\#}(B) \oplus \mathcal{P}(F)$ be a polarization on E, and $\Gamma_c(\mathcal{V})$ the subspace of sections covariant constant along $\mathcal{P}(E)$. Then $\Psi \in \Gamma_c(\mathcal{V})$ if $v\Psi = i\langle \alpha_B, v\rangle \Psi + \Psi A(v)$ for all $v \in \mathcal{D}(B)$.*

In this manner, the covariant constant condition is given in terms of the canonical one form on B, together with the connection A. In general it is not true that a

polarization $\mathcal{P}(B)$ on (B, Ω_B) will result in a polarization $\mathcal{P}^{\#}(B) \oplus \mathcal{P}(F)$ on E. In vector bundle quantization, one has to deal with the given symplectic form on B, and the hidden symplectic form on the total space E which induce a connection A on the bundle. In multi-component WKB theory one faces a similar situation with two symplectic forms [4].

In our formulation we have imposed two rather strong technical conditions; that the transition functions preserve the canonical form α_F and the polarization $\mathcal{P}(F)$, and that the commutator between the horizontal vector fields and $\mathcal{P}(F)$ remains in $\mathcal{P}(F)$.

The conditions on transition functions are not entirely necessary, it allows us to keep the technical details to a minimum, in that the quantization of the symplectic transforms $\chi_{ij,b}$ is given by (3). Quantization of non-polarization preserving symplectomorphisms can be obtained through the use of BKS-pairing [5].

Condition (5) can be considered as a "minimal coupling" condition. In the standard Dirac quantization, the momentum variable is quantized as $p \mapsto \frac{\mathrm{d}}{\mathrm{d}q} + A$ where A is a vector potential. This assignment presumes an interplay between the polarization (in this case the vertical polarization so that the wavefunctions are functions on the configuration space) and the connection A. Our assumption (5) represents an interrelation of this kind.

III. Particles in an external Yang-Mills field.

The case of Yang-Mills field can be described as follows [6]; Let $N \to Q$ be a principal bundle with group G and connection α_{YM}, the Yang-Mills potential, there is an $\alpha_{\mathrm{YM}}-$dependent projection $T^*N \to T^*Q$. The $G-$action on N can be lifted to a Hamiltonian $G-$action on T^*N with moment map $J : T^*N \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ such that the coadjoint orbit $\mathcal{O}_\mu$ is integral, and $E = J^{-1}(\mathcal{O}_\mu)/G \sim J^{-1}(\mu)/H$ be the Marsden-Weinstein reduced space [7], where H its isotropy subgroup of μ. Let $B = T^*Q$, then $E \to B$ is a fiber bundle whose fiber is $F = \mathcal{O}_\mu$, the transition function is induced by $G-$action on F, which preserves the canonical one–form on $\mathcal{O}_\mu$. Moreover, there is a standard polarization, the positive Kähler polarization, which is $G-$invariant. Quantization of F with respect to this polarization gives a irreducible representation space $\mathcal{Q}(F)$ of G induced by the $U(1)$ representation of H [8]. These are interpreted physically as the internal symmetries of a particle of "charge" μ in the configuration space Q. The vertical polarization on T^*Q lifts to a distribution on E satisfying condition (5). Thus all the assumptions we laid out in the previous section hold.

Denote by $N^{\#}$ the principal $G-$bundle over T^*Q by the pullback

$$\begin{array}{ccc} N^{\#} & \longrightarrow & N \\ \downarrow & & \downarrow \\ T^*Q & \longrightarrow & Q \end{array}$$

There is a diffeomorphism $N^{\#} \to J^{-1}(\mu) \subset T^*N$ given by

$$\xi = \pi^*(n)p + \mu \cdot \alpha_{\mathrm{YM}}(n) \tag{8}$$

where $\xi \in T^*_n N$, $p \in T^*_{\pi(n)}Q$, $\alpha_{\mathrm{YM}}(n) \in T^*_n N \otimes \mathfrak{g}$ and the dot product refers to the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. With these, the prequantization vector bundle on T^*Q can be conveniently described as

$$\mathcal{V} = N^{\#} \times \mathcal{Q}(F) \to T^*Q$$

Fix $b = (p, q) \in B$, and $v \in T_b B$, there is a natural projection $\Pi : T_b B \to T_q Q$. Let $\chi_{_i} : Q \to N$ be a local coordinate patch. In this setting, $\alpha_E - \pi^* \alpha_B = \mu \cdot \alpha_{_{\rm YM}}$ and the function $w = \langle \chi_{_i}^* \alpha_\nabla, v\rangle : \mathcal{O}_\mu \to \mathbf{R}$ becomes

$$w(gH) = \mu \cdot Ad_{g^{-1}} \langle \chi_{_i}^* \alpha_{_{\rm YM}}(\chi_{_i}), \Pi(v)\rangle,$$

where we have identified $\mathcal{O}_\mu = G/H$. Let $\mathfrak{u}$ be the Lie-algebra of unitary transforms on $\mathcal{Q}(F)$ and ρ the Lie-algebra representation $\rho : \mathfrak{g} \to \mathfrak{u}$. Then one can show that the connection A defined in (7) is given by

$$A^i(b)(v) = \rho \langle \chi_{_i}^* \alpha_{_{\rm YM}}(\chi_{_i}), \Pi(v)\rangle \tag{9}$$

The Yang-Mills potential $\alpha_{_{\rm YM}}$ also plays a cruicial role in the quantization of observables on T^*Q. In particular, the kinetic energy of the particle is quantized to $\frac{1}{2}(-\Delta_\alpha + R/6)$ where Δ_α is the covariant Laplace-Beltrami operator with respect to $\alpha_{_{\rm YM}}$, and R is the Ricci scalar curvature of the Riemannian manifold Q [9].

IV. Proof of theorem.

Our assertion will be proved with a sequence of propositions. We must first show that the inner product in (7) is well defined, recall $\alpha^i = \chi_{_i}^* \alpha_\nabla$,

Propostion 2. *If $\phi \in \mathscr{L}(F)$ is covariant constant along $\mathcal{P}(F)$, then $\mathscr{O}(\langle \alpha^i, v\rangle)\phi$ is also covariant constant along $\mathcal{P}(F)$. Thus $\mathscr{O}(\langle \alpha^i, v\rangle) : \mathcal{Q}(F) \to \mathcal{Q}(F)$.*

Once we have established that A^i is well defined, we must then show that A^i transforms like a gauge:

Proposition 3. *If $b \in U_i \cap U_j$ then $A^j(b) = \mathbf{X}_{ij,b} A^i(b) \mathbf{X}_{ij,b}^{-1} + d\mathbf{X}_{_{ij,b}} \mathbf{X}_{_{ij,b}}^{-1}$.*

Lastly we will show that the connection A has the desired property:

Propostion 4. *Let $\Psi \in \Gamma(\mathcal{V})$ and $\psi = \Psi_\nu \phi_\nu$ be the corresponding section in $\mathscr{L}$, let $v \in D(B)$. If $v^\# \psi = i\langle \alpha_E, v^\#\rangle \psi$, then $v(\Psi) = \Psi\,(i\langle \alpha_B, v\rangle I + A(v))$, where I is the identity matrix so that $i\langle \alpha_B, v\rangle I \in \mathfrak{u}$.*

Proof of Proposition 2. Let $\phi \in \mathscr{L}(F)$ covariant constant along $\xi \in \mathcal{P}(F)$, then

$$\xi\phi = i\langle \alpha_F, \xi\rangle \phi. \tag{10}$$

For any fixed $b \in U_i$ consider the operator $\mathscr{O}(w) = -i\mathscr{H}_w - \langle \alpha_F, \mathscr{H}_w\rangle + w$ where $w = \langle \alpha^i, v\rangle$ is treated as a function on F. Then

$$\begin{aligned} \xi\mathscr{O}(w)\phi - i\langle \alpha_F, \xi\rangle \mathscr{O}(w)\phi = {} & (w - \langle \alpha_F, \mathscr{H}_w\rangle)\,(\xi\phi - i\langle \alpha_F, \xi\rangle\phi) \\ & - i\xi\,(\mathscr{H}_w \phi) - (\xi\langle \alpha_F, \mathscr{H}_w\rangle)\,\phi - \langle \alpha_f, \xi\rangle \mathscr{H}_w \phi + (\xi w)\,\phi \end{aligned} \tag{11}$$

Here the right hand side of first line in (11) vanishes because of (10). Using (2) we have

$$\xi\langle \alpha_F, \mathscr{H}_w\rangle\phi = (\mathscr{H}_w \langle \alpha_F, \xi\rangle)\phi - \langle \alpha_F, [\mathscr{H}_w, \xi]\rangle\phi + (\xi w)\phi.$$

Equation (11) then becomes

$$\begin{aligned} & -i\xi(\mathscr{H}_w\phi) - \mathscr{H}_w(\langle \alpha_F, \xi\rangle)\,\phi + \langle \alpha_F, [\mathscr{H}_w, \xi]\rangle\phi - \langle \alpha_F, \xi\rangle \mathscr{H}_w \phi \\ = {} & -i\xi(\mathscr{H}_w\phi) - \mathscr{H}_w(\langle \alpha_F, \xi\rangle\phi) + \langle \alpha_F, [\mathscr{H}_w, \xi]\rangle\phi \\ = {} & -i\xi(\mathscr{H}_w\phi) + i\mathscr{H}_w(\xi\phi) + \langle \alpha_F, [\mathscr{H}_w, \xi]\rangle\phi \\ = {} & i\,([\mathscr{H}_w, \xi]\phi - i\langle \alpha_F, [\mathscr{H}_w, \xi]\rangle\phi) \\ = {} & 0 \end{aligned}$$

since $v - \mathscr{H}_w$ is horizontal by proposition 1, and assumption (5) implies $[v - \mathscr{H}_w, \xi] = -[\mathscr{H}_w, \xi] \in \mathcal{P}(F)$. $\square$

Proof of proposition 3. Here we show A defines a connection on the vector bundle $\mathcal{V}$. For $b \in U_i \cap U_j$ and $v \in T_b B$, we let $[A^i(b)v]_{\mu\nu} = i \ll \phi_\nu \,|\, \mathscr{O}(w^i)\phi_\mu \gg$ and $[A^j(b)v]_{\mu\nu} = i \ll \phi_\nu \,|\, \mathscr{O}(w^j)\phi_\mu \gg$ where $w^i(b,f) = \langle \alpha^i(b,f), v \rangle$, $w^j(b,\bar{f}) = \langle \alpha^j(b,\bar{f}), v \rangle$ with $\chi_i(b,f) = \chi_j(b,\bar{f}) = e$, so $\bar{f} = \chi_{ij,b} f$, and $\alpha^i(b,f) = \chi_i^* \alpha_\nabla(e)$, $\alpha^j(b,\bar{f}) = \chi_j^* \alpha_\nabla(e)$. We calculate

$$\begin{aligned} w^i(b,f) &= \langle \chi_i^* \alpha_\nabla(e), v \rangle = \langle \chi_{ij}^* \alpha^j(b,\bar{f}), v \rangle = \langle \alpha^j(b,\bar{f}), \chi_{ij,*} v \rangle \\ &= \langle \alpha^j(b,\bar{f}), v + J_* v \rangle = w^j(b,\bar{f}) + \langle \alpha^j(b,\bar{f}), J_* v \rangle \\ &= w^j(b,\bar{f}) + \langle \alpha_F(\bar{f}), J_* v \rangle \end{aligned} \tag{12}$$

where J_* is the block matrix in

$$\chi_{ij*} = \begin{bmatrix} I & 0 \\ J_* & \chi_{ij,b*} \end{bmatrix} : T_b B \times T_f F \to T_b B \times T_{\bar{f}}, F$$

so that $J_* v \in T_{\bar{f}} F$ is a vertical vector. Denote the last term in (12) by $\beta(b,f)$, then we have $w^j(b,\bar{f}) = w^i(b,f) - \beta(b,f)$. Writing out explicitly the dependence on F

$$\begin{aligned} [A^j(b)v]_{\mu\nu} &= i \ll \phi_\nu(\bar{f}) \,|\, \mathscr{O}(w^j(b,\bar{f})\phi_\mu(\bar{f}) \gg \\ &= i \ll \phi_\nu(\bar{f}) \,|\, \mathscr{O}(w^i(b,f)\phi_\mu(\bar{f}) \gg - i \ll \phi_\nu(\bar{f}) \,|\, \mathscr{O}(\beta(b,f))\phi_\mu(\bar{f}) \gg \end{aligned} \tag{13}$$

The first term in (13) reduces to

$$\begin{aligned} i \ll \phi_\nu(\bar{f}) \,|\, \mathscr{O}(w^i(b,f)\phi_\mu(\bar{f}) \gg &= i \ll \mathbf{X}_{\nu\tau}\phi_\tau(f) \,|\, \mathscr{O}(w^i(b,f)\mathbf{X}_{\mu\sigma}\phi_\sigma(f) \gg \\ &= \overline{\mathbf{X}}_{\nu\tau}\mathbf{X}_{\mu\sigma}[A^i(b)v]_{\sigma\tau} = \mathbf{X}_{\mu\sigma}[A^i(b)v]_{\sigma\tau}\overline{\mathbf{X}}^t_{\tau\nu} \end{aligned} \tag{14}$$

which gives the familiar adjoint action $\mathbf{X}A^i\mathbf{X}^{-1}$ in matrix notation, here $\mathbf{X} = \mathbf{X}_{ij,b}$ as in (3).

As for the second term in (13), suppose we introduce local canonical coordinates (p,q) around f and $(\overline{p},\overline{q})$ around $\bar{f}$, so that $(\overline{p}(p,q,b), \overline{q}(p,q,b)) = \chi_{ij,b}(p,q)$, and $v\overline{p}$ is the vector field v operating on $\overline{p}$ as a function on B. In these coordinates,

$$\beta(b,f) = \langle \alpha_F(\bar{f}), J_* v \rangle = \overline{p}_m v(\overline{q}_m) \tag{15}$$

$$\mathscr{H}_\beta = \left[\frac{\partial \overline{p}_m}{\partial p_n} v(\overline{q}_m) + \overline{p}_m \frac{\partial v(\overline{q}_m)}{\partial p_n} \right] \frac{\partial}{\partial q_n} - \left[\frac{\partial \overline{p}_m}{\partial q_n} v(\overline{q}_m) + \overline{p}_m \frac{\partial v(\overline{q}_m}{\partial q_n} \right] \frac{\partial}{\partial p_n}$$

Since $\chi_{ij,b}$ preserves the canonical one–form α_F, we have

$$\overline{p}_m \frac{\partial \overline{q}_m}{\partial p_n} = p_n, \qquad \overline{p}_m \frac{\partial \overline{q}_m}{\partial p_n} = 0,$$

$$\{\overline{p}_m, \overline{q}_n\} = \delta_{mn}, \qquad \{\overline{p}_m, \overline{p}_n\} = \{\overline{q}_m, \overline{q}_n\} = 0 \tag{16}$$

where $\{,\}$ denotes the Poison bracket on F. Using (16), one can show that

$$\langle \alpha_F(f), \mathscr{H}_\beta \rangle = \overline{p}_m v(\overline{q}_m), \qquad \{\mathscr{H}_\beta, \overline{p}_m\} = v(\overline{p}_m), \qquad \{\mathscr{H}_\beta, \overline{q}_m\} = v(\overline{q}_m). \tag{17}$$

With (15) and (17), we get

$$\begin{aligned} -i\mathscr{O}(\beta(b,f))\phi_\mu(\bar{f}) &= \mathscr{H}_\beta \phi_\mu(\bar{f}) + i\left(\langle \alpha_F(f), \mathscr{H}_\beta\rangle - \beta\right)\phi_\mu = \mathscr{H}_\beta\phi_\mu(\bar{f}) \\ &= \frac{\partial \phi_\mu(\bar{f})}{\partial \overline{p}_m}\{\mathscr{H}_\beta, \overline{p}_m\} + \frac{\partial \phi_\mu(\bar{f})}{\partial \overline{q}_m}\{\mathscr{H}_\beta, \overline{q}_m\} \\ &= \frac{\partial \phi_\mu(\bar{f})}{\partial \overline{p}_m} v(\overline{p}_m) + \frac{\partial \phi_\mu(\bar{f})}{\partial \overline{q}_m} v(\overline{q}_m) = v(\phi_\mu(\bar{f})) \\ &= v[\mathbf{X}_{\mu\sigma}]\phi_\sigma(f) \end{aligned}$$

and the second term in (13) becomes

$$\begin{aligned} -i \ll \phi_\nu(\bar{f}) \,|\, \mathscr{O}(\beta(b,f))\phi_m u(\bar{f}) \gg &= \ll \mathbf{X}_{\nu\tau}\phi_\tau(f) \,|\, v[\mathbf{X}_{\mu\sigma}]\phi_\sigma(f) \gg \\ &= \overline{\mathbf{X}}_{\nu\tau} v[\mathbf{X}_{\mu\tau}] = v[\mathbf{X}_{\mu\tau}]\overline{\mathbf{X}}^t_{\tau\mu} \end{aligned}$$

which in matrix notation is $v\mathbf{X}\mathbf{X}^{-1}$. $\square$

Proof of proposition 4. Let $\psi = \Psi_\mu\phi_\mu$ be a section on $\mathscr{L}$ covariant constant along $v^\#$, let $w = \langle \alpha^i, v\rangle$, then on $U_i \times F$ we have

$$\begin{aligned} (v - \mathscr{H}_w)\Psi_\mu\phi_\mu &= i(\langle \alpha^i, v\rangle + \langle \alpha_B, v\rangle - \langle \alpha_F, \mathscr{H}_w\rangle)\Psi_\mu\phi_\mu \\ (v\Psi_\mu)\phi_\mu - \Psi_\mu(\mathscr{H}_w\phi_\mu) &= iw\Psi_\mu\phi_\mu + i\langle \alpha_B, v\rangle\Psi_\mu\phi_\mu - i\langle \alpha_F, \mathscr{H}_w\rangle\Psi_\mu\phi_\mu \\ (v\Psi_\mu)\phi_\mu &= i\left(-i\mathscr{H}_w\phi_\mu - \langle \alpha_F, \mathscr{H}_w\rangle\phi_\mu + \langle \alpha_B, v\rangle\phi_\mu + w\phi_\mu\right)\Psi_\mu \\ &= i(\langle \alpha_B, v\rangle\phi_\mu + \mathscr{O}(w)\phi_\mu)\Psi_\mu \end{aligned}$$

by (6). Thus

$$\begin{aligned} \ll \phi_\nu \,|\, (v\Psi_\mu)\phi_\mu \gg &= i \ll \phi_\nu \,|\, \langle \alpha_B, v\rangle\phi_\mu + \mathscr{O}(w)\phi_\mu \gg \Psi_\mu \\ v\Psi_\nu(b) &= \Psi_\mu(b)\left(i\langle \alpha_B, v\rangle\delta_{\mu\nu} + \left[A^i(b)v\right]_{\mu\nu}\right) \end{aligned}$$

according to the definition (7). $\square$

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